

High-resolution electromagnetic imaging of the conductive Earth interior

Dominique Gibert†§, Benoît Tournier†|| and Jean Virieux†¶

† Géosciences Rennes, Université Rennes 1, Campus de Beaulieu, 35042 Rennes cedex, France

‡ Institut de Géodynamique, CNRS, Avenue A Einstein, 06560 Valbonne, France

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Abstract. Ohmic dissipation in conductive media considerably limits the penetrative power of high-frequency electromagnetic imaging methods and implies that deep regions can be probed only with low-frequency fields. Unfortunately, these low-frequency fields are governed by a diffusive equation which prevents direct high-resolution imaging as in seismic and georadar imaging. However, a clue for high-resolution imaging in the diffusive approximation is given by a Fredholm integral equation of the first kind which links diffusive fields to their propagative duals. If these duals could be recovered by inverting this integral equation, the seismic imaging toolbox might be used, at least from a theoretical point of view, to produce fine electromagnetic images. Spectral decomposition of the integral operator shows that the inverse problem is numerically ill-posed for both noisy and/or incomplete data. High-resolution can be achieved only by adding sparsity constraints upon the sought solution to the information content of the data. This type of *a priori* information also strongly regularizes the inversion but implies that the inverse problem must be treated as non-linear. A numerical algorithm, designed to work in a continuous parameter space, couples both the simulated annealing and the simplex to recover the propagative field. Numerical applications for pseudo-data with additive noise reveal that reflective interfaces can be imaged even within the poorly-favourable magnetotelluric setup.

1. Introduction

Imaging the Earth's interior is a major task in geophysics which involves different techniques

from the fact that rocks are electrical, conductive media inside which the electromagnetic field is governed by the telegrapher equation,

$$\nabla \times \nabla \mathbf{E}(\mathbf{r}, t) + \mu \varepsilon(\mathbf{r}) \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) + \mu \sigma(\mathbf{r}) \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) = \mathbf{S}(\mathbf{r}, t) \quad (1)$$

where μ , ε and σ stand for the magnetic permeability, and the electric permittivity and conductivity respectively. $\mathbf{S}(\mathbf{r}, t)$ represents a source term. General solutions to this equation are damped waves but in the Earth, where the conductivity is large, the wave-like character of the electromagnetic field disappears very quickly to the benefit of the diffusive behavior [29] and, for most geophysical models, the electromagnetic field is given by the diffusion equation,

$$\nabla \times \nabla \mathbf{E}(\mathbf{r}, t) + \mu \sigma(\mathbf{r}) \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) = \mathbf{S}(\mathbf{r}, t). \quad (2)$$

High-resolution electromagnetic imaging in this diffusive context has long been considered as an impossible challenge, mainly for the theoretical reason that diffusion constitutes a formal barrier to any ‘diffraction-like’ tomography. Links exist, however, between diffusion and propagation [9, 17, 27] and their study could cast new insights upon the possibilities for high-resolution electromagnetic imaging in a diffusive context. The need for a detailed study of the transformation from a diffusive to a propagative field is further motivated by the fact that the state of the art of wave-field imaging, which is now well developed [3], might be used to the benefit of diffusive-field imaging if a bridge could be put between the two domains [38]. For instance, and to illustrate this strategy, it is well established that wave-field imaging does not allow for high-resolution reconstructions of the velocity distribution but does for the reflectivity (i.e. impedance gradients) [13]. This result is also valid for the diffusive case and indicates that imaging in terms of the conductivity distribution (the electromagnetic equivalent of seismic velocity) will not allow for a high resolution which could be reached only in terms of conductivity gradients (i.e. the equivalent of seismic reflectivity). This strongly guides the way to parametrize the inverse problem. The goals of this paper are both to precisely document the formal inversion from a diffusive to a propagative field and, then, to rely on this formal inverse to set up a practical numerical inversion which could operate on actual data. Section 2 documents the links between propagation and diffusion and solves the inverse problem of transforming a diffusive field into a propagative dual. Section 3 examines and discusses the various constraints we use to stabilize the inversion of incomplete and noisy data. A numerical solution using simulated annealing is presented in section 4, and realistic synthetic examples are presented in section 5. Although primarily designed for geophysical purposes this study also applies to other fields interested in imaging conductive media such as in non-destructive control and medical imaging [41, 42].

2. From a diffusive to a propagative field

To set a link between diffusion and propagation we follow the approach given in [19] which formally introduces a field satisfying the wave equation

$$\nabla \times \nabla U(\mathbf{r}, q) + \mu \sigma(\mathbf{r}) \frac{\partial^2}{\partial q^2} U(\mathbf{r}, q) = \mathbf{F}(\mathbf{r}, q) \quad (3)$$

where q is an independent variable playing the role of pseudo-time with a physical dimension of square-root of time, and $F(\mathbf{r}, q)$ is a source term. The wave speed $c(\mathbf{r}) = 1/\sqrt{\mu\sigma(\mathbf{r})}$. An integral equation between $E(t)$ and $U(q)$,

$$E(t) = \int_0^\infty \frac{q}{2\sqrt{\pi t^3}} \exp\left(-\frac{q^2}{4t}\right) U(q) dq \quad (4)$$

introduces a kernel corresponding to the three-dimensional Green function of the diffusion equation. The spatial dependence of the fields has been left implicit since the transformation only involves the time t and the variable q . The source terms $S(\mathbf{r}, t)$ and $F(\mathbf{r}, q)$ are related by the same expression. The direct problem of transforming a propagative field into a diffusive one is numerically stable as one may expect from the damped nature of the integral kernel in (4), and the inverse transformation from a diffusive field $E(\mathbf{r}, t)$ into its propagative dual $U(\mathbf{r}, q)$ it must be stressed is highly unstable. This is more understandable in the Fourier domain ($t \rightarrow \nu$) where (4) transforms into

$$E(\nu) = \int_0^\infty \exp(-q\sqrt{i\nu}) U(q) dq. \quad (5)$$

This Fredholm equation possesses the nice property that its kernel only depends upon the

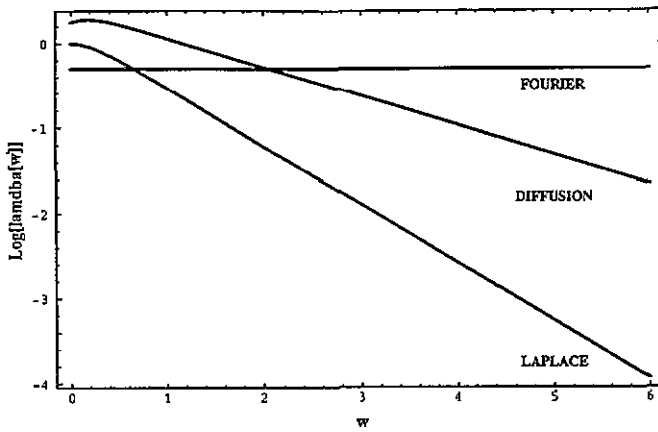


Figure 1. Eigenvalue spectra for different integral operators. The constant spectrum is for the Fourier transform while the most-decreasing one is for the Laplace transform. The spectrum labelled 'diffusion' corresponds to (9).

Provided that $E(\nu)$ is noiseless and known for every $\nu \geq 0$, (10) gives the complete solution to our inverse problem. The eigenvalues λ_{ω}^{\pm} quantify the efficiency by which their corresponding eigenfunctions are transmitted into the data (see (6)): the larger the eigenvalue the more efficient the transmission of the information element. Eigenfunctions whose eigenvalues are too small become lost in the noise, and the process of dividing by such small eigenvalues in (10) will dramatically amplify any noise present in the data. Unfortunately, the spectrum (9) decays asymptotically to zero as $\omega \rightarrow \infty$ (figure 1), and this makes reconstruction (10) ill posed. Including the high-mode eigenfunctions in the solution will produce meaningless results, and our solution may be rewritten as the sum of two components,

$$\begin{aligned}
 U(q) &= \int_0^{\omega_{\max}} d\omega \frac{\Phi_{\omega}^{\pm}(q)}{\lambda_{\omega}^{\pm}} \int_0^{\infty} E(\nu) \Phi_{\omega}^{\pm}(\sqrt{\nu}) d\nu + \int_{\omega_{\max}}^{\infty} d\omega \frac{\Phi_{\omega}^{\pm}(q)}{\lambda_{\omega}^{\pm}} \int_0^{\infty} E(\nu) \Phi_{\omega}^{\pm}(\sqrt{\nu}) d\nu \\
 &= U_d(q) + U_n(q)
 \end{aligned} \tag{11}$$

where the cutoff ω_{\max} depends on the signal-to-noise ratio. The components $U_d(q)$ and $U_n(q)$ correspond respectively to the part of the solution which can be recovered from the data and to the one which cannot be. $U_d(q)$ is a low-pass filtered version of the true solution $U(q)$ and is insufficient to allow for a high-resolution reconstruction. In fact, the resolution follows the distribution of the zero-crossings of $\Phi_{\omega_{\max}}^{\pm}$ and rapidly deteriorates when q increases (figure 2). The only way to improve the resolution is to perform the inversion not only with data but also with some *a priori* information [34] whose role will be to extrapolate the reconstruction beyond the cutoff ω_{\max} . From the point of view of inverse problem theory, this extrapolation may be considered as using *a priori* information to complete the basic solution $U_d(q)$ with a non-null $U_n(q)$ constructed in the null space generated by the basis $\{\Phi_{\omega}^{\pm}, \omega > \omega_{\max}\}$.

3. Constraints and parametrization

The *a priori* information suitable to our inverse problem are of three types. The first constraint concerns the causality of the field $U(q)$ and limits the support of the solution to \mathbb{R}^+ .

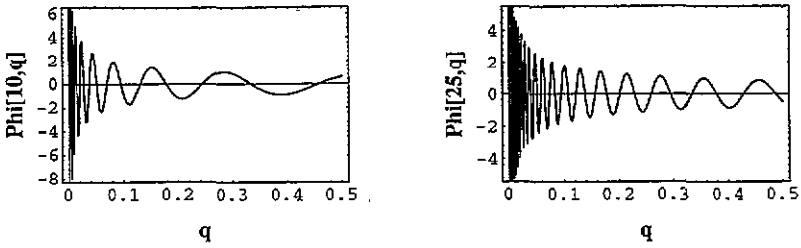


Figure 2. $\Phi_{\omega}^+(q)$ for $\omega = 10$ and $\omega = 25$.

A second constraint comes from the objectives of imaging which are of structural concern. In other words, the goal is to detect the edges of the more or less numerous homogeneous blocks which constitute the media to probe. This explains why the reflectivity is the right parameter to seek, since it is directly related to the edges of the structures where most of the jumps of the electromagnetic properties of the body occur. By definition these edges form a very sparse and irregular lattice in space which, thanks to the Huygens principle, will back-scatter sparse wavefronts when visited by an incident wavefront. As a consequence, and assuming an impulsive source term, $F(q)$, the sequences, $U(q)$, recorded at each receptor will be sparse trains of replicas of this initial short pulse (see numerical simulations in [37]). This will be true only for real reflection coefficients and, equivalently, if the incident wavefront attacks the edges of the structures under subcritical conditions. This allows the success of the seismic method under near-normal reflection conditions. Were these conditions to be violated, several reflection coefficients might become complex, and the corresponding reflected events would no longer be simple replicas of the source term [1]. In the remaining part of this discussion we shall assume that all reflection coefficients are real and that

$$U(q) = \sum_{n=1}^N U_n F(q - q_n) \tag{12}$$

is verified. Such a sparsity constraint upon the unknown field has been recognized to strongly stabilize ill-posed inverse problems similar to the present one [6]. Indeed, such a constraint has already successfully been applied to deconvolution of noisy seismic data [21]. The last constraint acts upon the nature of the source terms. If the electromagnetic source is known or if it is possible to deconvolve the data, which is merely the case in magnetotellurics, we may assume a Dirac impulse for the initial pulse $S(t)$. The corresponding propagative source term, $F(q)$, is also a Dirac impulse as is readily seen from (5) and (12) becomes

$$U(q) = \sum_{n=1}^N U_n \delta(q - q_n) \tag{13}$$

which, when inserted into (5), gives

$$E(v) = \sum_{n=1}^N U_n \exp(-q_n \sqrt{i\nu}) \tag{14}$$

In practice, the data are known for a finite set of discrete frequencies $\{E_{\text{data}}(\nu_1), E_{\text{data}}(\nu_2), \dots, E_{\text{data}}(\nu_M)\}$ from which one has to get estimates of the unknown parameters $\{N, q_1, q_2, \dots, q_N, U_1, U_2, \dots, U_N\}$. This is the actual inverse problem which is discussed in the following section.

4. Computer implementation by simulated annealing

The inverse problem defined in the preceding section is linear with respect to the U_i s and highly non-linear with respect to both the q_i s and N as seen in (14). Provided a suitable frequency sampling model (14) can be recast under the form of the Prony model

any tunnelling effect which could allow the trapped model to escape from the local minimum. Conversely, a too slow decrease may result in prohibitive computational costs. Various cooling schedules have been proposed [16], and several studies [28,25] about practical implementation of the simulated annealing algorithm favour the cooling schedule [12,33]

$$T(n+1) = T(n) \exp\left(-\frac{\lambda T(n)}{\text{sd}(S(n))}\right) \quad (16)$$

where λ is a parameter called the thermodynamic speed, and $\text{sd}(S(n))$ is the standard deviation of the costs of the models accepted during the Metropolis algorithm at temperature $T(n)$. The choice of the value for λ is not crucial and we retain $\lambda = 0.75$. This cooling schedule is adaptive in the sense that the decrease of the temperature is slower when the models accepted during the preceding Metropolis loops have almost identical costs. In doing so, the schedule reiterates a sequence of Metropolis loops at an only-slightly lowered temperature such that the probability for an escape from an eventual local minimum is kept high. The initial temperature in this cooling schedule must be taken sufficiently high to allow for a wide exploration of the parameter space and the final temperature is set to one hundredth of the noise variance [10].

The second question to answer when implementing the algorithm is: how to perturb the current model to obtain the new model to be tested? Most applications of simulated

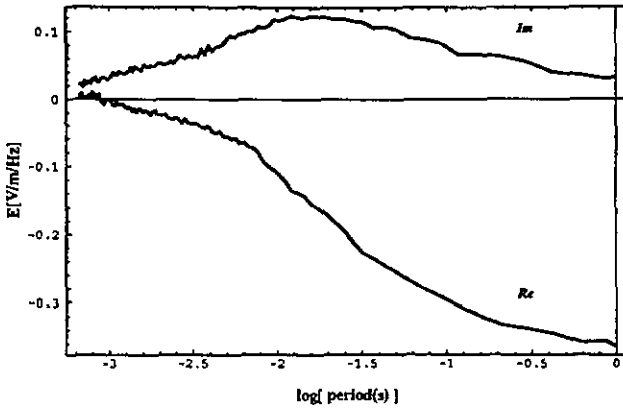


Figure 3. Pseudo-data for a station ($x = 750$ m) of the *en biseau* model (top of figure 4).

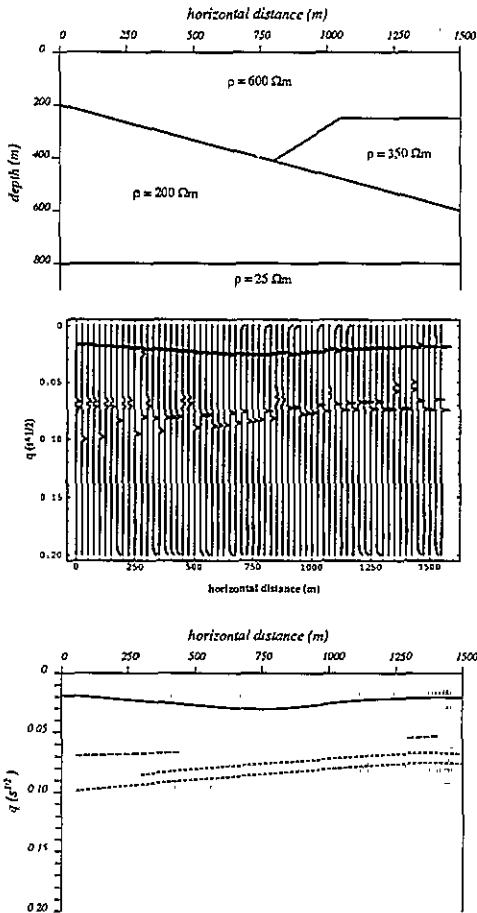


Figure 4. *En biseau* model. Top: sketch of the structure from which pseudo-data are computed. Middle, conditional probabilities of the inverted reflector. Bottom: main reflectors picked from the conditional-probability section.

of this algorithm follows what we described in our previous paper, namely a hierarchical determination of the parameters and a jump of the specific heat when approaching a parameter determination (see [10] for more details).

Two inversions of 2D models have been done. The first model consists of an *en biseau* structure (top of figure 4) from which we generated pseudo-magnetotelluric data at 63

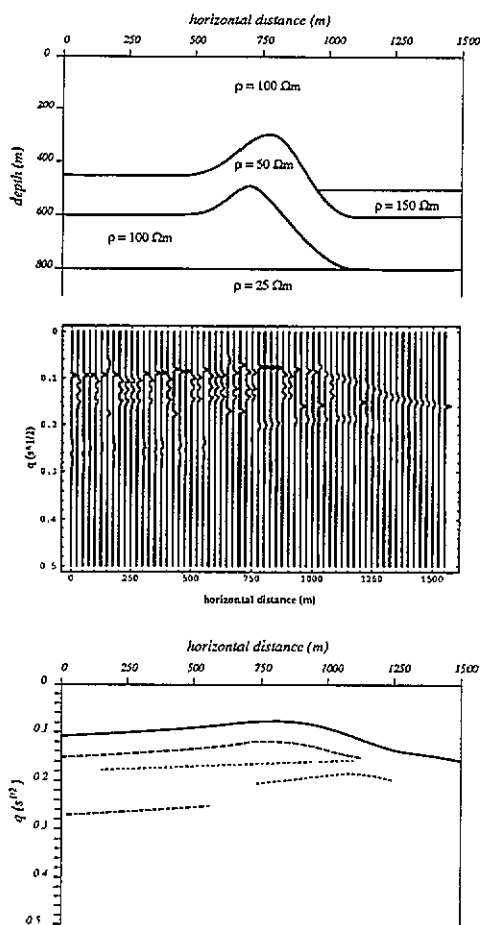


Figure 5. Anticlinal model. Top: sketch of the structure from which pseudo-data are computed. Middle: conditional probabilities of the inverted reflector. Bottom: main reflectors picked from the conditional-probability section.

equidistant stations along the surface. A finite-element program is used for this purpose [39]. The data for each station are computed for frequencies ranging from 1 to 1500 Hz with an additive white Gaussian noise ($\sigma^2 = 10^{-5}$) corresponding roughly to a signal-to-noise ratio of 30 dB (figure 2). Each trace is inverted independently and our inversion consists

figure 5): the shallowest reflector remains well-detected but the deeper ones are somewhat interlaced and do not allow for a clear unambiguous interpretation (bottom of figure 5). This partial failure of the inversion appeals for a true 2D inverse algorithm which could ensure coherency between adjacent traces.

6. Conclusion

We have shown that reflectivity electromagnetic imaging in a diffusive context is possible provided a careful parametrization of the inverse problem is used. This demonstrates the importance of *a priori* information such as sparsity constraints which, however, make the inverse problem highly non-linear. Use of simulated annealing coupled with the simplex algorithm allows for a rapid convergence without neglecting resolution. Synthetic assessments of the method with pseudo-data show that good results are obtained when the data are corrupted with a reasonable amount of noise. More stable results could be reached if the inversions of section-traces could be performed simultaneously by accounting for some coherency criteria. This next step is in progress and will be described in a future paper.

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